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An Inferentialist Approach to Paraconsistency

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Abstract
This paper develops and motivates a paraconsistent approach to semantic paradox from within a modest inferentialist framework. I begin from the bilateralist theory developed by Greg Restall, which uses constraints on assertions and denials to motivate a multiple-conclusion sequent calculus for classical logic, and, via which, classical semantics can be determined. I then use the addition of a transparent truth-predicate to motivate an intermediate speech-act. On this approach, a liar-like sentence should be "weakly asserted", involving a commitment to the sentence and its negation, without rejecting the sentence. From this, I develop a proof-theory, which both determines a typical paraconsistent model theory, and also gives us a nice way to understand classical recapture.

Introduction

This paper develops and motivates a paraconsistent approach to semantic paradox from within a modest inferentialist framework. There are many different forms of inferentialism. By modest inferentialism, I will mean a view on which a specified set of inference rules are taken to determine the truth-conditional content of logical constants, and where those rules have a substantive connection with ordinary inferential practices (Belnap and Massey, 1990; Boghossian, 2003; Garson, 2010; Peacocke, 1986a,b, 1987). Much of the existing work on paraconsistent logic emphasizes the construction of many-valued semantic consequence (Beall, 2013; Priest, 2006, 2008). By expanding upon the bilateralist theory of inferentialism, the aim is to develop a paraconsistent proof-theory that itself determines the model theory.

In §1, I outline the bilateralist framework developed by Greg Restall (Restall, 2005, 2009), which uses constraints on assertions and denials to motivate a multiple-conclusion sequent calculus for classical logic. I also show how classical semantics can be determined by the calculus. §2 introduces a transparent truth-predicate into the calculus, which, following advocates of non-classical logic (e.g Parsons 1984; Priest 2006), is taken to motivate a non-primitive attitude intermediate between assertion and denial. On this approach, a liar-like sentence should be "weakly asserted", involving a commitment to the sentence and its negation, without rejecting the sentence. §4 outlines a corresponding model-theory (broadly this is Beall's (2013) LP^+), before showing that any ordinary sequent calculus fails to completely determine that semantics. To deal with this, in §5, I outline a 3-sided proof-theory, which both determines the semantics LP^+, and also gives us a nice way to understand classical recapture in limit cases. Whilst there are a few novel technical results scattered throughout, the real novelty lies in the philosophical setting for a generalized program of paraconsistent modest inferentialism.

1 Modest inferentialism

This section briefly sketches the bilateralist approach to inferentialism. The modest version of inferentialism that we are interested in here allows that the meanings of logical constants in a language $\mathscr{L}$ are explained in terms of which inferences are valid in $\mathscr{L}$. §1.1 introduces the position and uses it to motivate a multiple-conclusion sequent calculus for classical logic. In a way that will be specified in §1.2, these inferences can be said to determine classical model-theory. Finally, in §1.3, the notion of absoluteness is introduced to characterize when a model-theory is completely determined by a sequent calculi.

1.1 Motivating classical sequent calculus

Logic tells us something about the way that agents' rational commitments are combined and constrained over arguments. For example, an agent asserting α, $\beta \vdash \alpha \wedge \beta$, may be said to be rationally committed to not simultaneously asserting α, β and denying $\alpha \wedge \beta$. Note that this way of putting things is deliberately inequivalent to saying that the agent asserting α, β is thereby rationally committed to asserting $\alpha \wedge \beta$. This is because, whilst such rational commitments play a key role in the fixation of beliefs, they neither commit an agent to logical omniscience, nor do they rationally oblige an agent to assert $\alpha \wedge \beta$ where, for example $\alpha \wedge \beta$ is unreasonable given the antecedent context of assertion. So, logic, on this view, constrains an agent's commitments by saying that it is rationally prohibitive to deny $\alpha \wedge \beta$, given the assertion of α, β.

Importantly, this view takes logical consequence to tell us not just about assertion, but about both assertion and denial, and the connection between the two. Restall's (2005; 2009) suggestion is to think of logical consequence as governing positions involving asserted and denied statements.

Definition 1. (Position) A position $[\Gamma : \Delta]$ is a pair of sets of formulae where Γ is the set of asserted formulas, and Δ the set of denied formulas.

A position expressed in a language may be used to represent an agent's rational commitments in terms of the coherence between assertions and denials. Where $[\Gamma : \Delta]$ is a position, we allow that $[\Gamma, \alpha : \Delta, \beta]$ is the state adding the formula α to the left set Γ, and β to Δ. Think of the above coherence constraints over rational commitment as saying that, a position $[\Gamma : \Delta]$ is incoherent if it contains some formula in both the left set and the right set, so that $\Gamma \cap \Delta \neq \emptyset$. Thinking of this in terms of an agent, such a position indicates that some statement is both asserted and denied, and so incoherent. Incoherence allows us to characterize sequent provability.

Definition 2. (Sequent provability) If $[\alpha : \beta]$ is incoherent, then $\alpha \vdash \beta$.

This follows because, if a position consisting of asserting α and denying β is incoherent, then $\alpha \vdash \beta$, and an agent who asserts α and denies β, as we said above, has made a mistake.

The definition generalizes in cases involving sets of assertions and denials. In a multiple-conclusion (SET-SET) framework, $\Gamma \vdash \Delta$ may be read in terms of the underlying atomic formulae $\{\alpha_1, \ldots, \alpha_n\} \vdash \{\beta_1, \ldots, \beta_n\}$, which is (classically) equivalent to $\{\alpha_1 \wedge \alpha_2 \ldots . \wedge \alpha_n\} \rightarrow \{\beta_1 \vee \beta_2 \ldots . \vee \beta_n\}$. Then, any position $[\Gamma : \Delta]$ for which an agent who asserts each member of Γ, and denies each member of Δ is incoherent. In that case, $\Gamma \vdash \Delta$, and an agent is mistaken to assert all $\alpha \in \Gamma$ and deny all $\beta \in \Delta$.

Definition 3. (Sequent provability generalized) If $[\Gamma : \Delta]$ is incoherent, then $\Gamma \vdash \Delta$.

The general idea is to build-up a sequent calculus out of these constraints over assertions and denials. First, consider the addition of structural constraints. Since both asserting and denying the same formula is incoherent, from $[\Gamma, \alpha : \Delta, \alpha]$ and Definition 3, we have the usual identity axiom for sequent-calculi.

$$\alpha \vdash \alpha \text{ for all } \alpha \text{ (Identity)}$$

We also have weakening, since, if a position is incoherent, the addition of assertions or denials will not bring it back to a coherent position. Contra-positively, if $[\Gamma : \Delta]$ is coherent, and $\Gamma' \subseteq \Gamma$ and $\Delta' \subseteq \Delta$, then $[\Gamma' : \Delta']$ will be coherent. This gives us:

$$\frac{\Gamma \vdash \Delta}{\Gamma, \Gamma' \vdash \Delta, \Delta'} \text{ (Weakening)}$$

Think of extensibility constraints on assertions and denials. For a position $[\Gamma : \Delta]$ is coherent, if the positions $[\Gamma, \alpha : \Delta]$ or $[\Gamma : \Delta, \alpha]$ are incoherent, then the original position $[\Gamma : \Delta]$ must already be incoherent. In other words, if a position is coherent, it should be extensible by a formula α to a coherent position where α is either asserted or α is denied. So, where $[\Gamma : \Delta]$ is coherent, either $[\Gamma, \alpha : \Delta]$ or $[\Gamma : \Delta, \alpha]$ is coherent. This gives us:

$$\frac{\Gamma, \alpha \vdash \Delta \qquad \Gamma \vdash \alpha, \Delta}{\Gamma \vdash \Delta} \text{ (Cut)}$$

Operational rules for the connectives can also be constructed fairly naturally out of positions. For example, if the position $[\Gamma : \Delta, \alpha \wedge \beta]$ is coherent, then $[\Gamma : \Delta, \alpha]$, $[\Gamma : \Delta, \beta]$, or both, are coherent. Contra-positively, if $[\Gamma : \Delta, \alpha]$ and $[\Gamma : \Delta, B]$ are incoherent, then so to is $[\Gamma : \Delta, \alpha \wedge \beta]$. In this case, we know that $\Gamma \vdash \Delta, \alpha$, and $\Gamma \vdash \Delta, \beta$, so that $\Gamma \vdash \Delta, \alpha \wedge \beta$. This gives us:

$$\frac{\Gamma, \alpha, \beta \vdash \Delta}{\Gamma, \alpha \wedge \beta \vdash \Delta} \wedge\text{-L} \qquad \frac{\Gamma \vdash \Delta, \alpha \qquad \Gamma \vdash \Delta, \beta}{\Gamma \vdash \Delta, \alpha \wedge \beta} \wedge\text{-R}$$

Keep in mind that, on this bilateralist approach, the meanings of connectives are built up from the primitive speech acts of both assertion and denial, in contrast to the assumption that denial of α is simply the assertion of $\neg\alpha$. We construct the rules for classical negation by taking a negation $\neg\alpha$ to be assertible when α is deniable, and vice-versa. So, if $[\Gamma : \Delta, \alpha]$ is incoherent, then so too is $[\Gamma, \alpha : \Delta]$. This gives us Gentzen's classical negation rules:

$$\frac{\Gamma \vdash \alpha, \Delta}{\Gamma, \neg\alpha \vdash \Delta} (\neg\text{-L}) \qquad \frac{\Gamma, \alpha \vdash \Delta}{\Gamma \vdash \neg\alpha, \Delta} (\neg\text{-R})$$

Analogous accounts can be provided for all of the classical sequent rules (Restall, 2005). This gives us a construction of the classical sequent rules in multiple-conclusion form, which is built out of a simple and plausible account of agents' rational commitments.

1.2 A determination theory

Given that we are working toward a modest inferentialism, we can think of these rules as also determining the truth-conditional content of the connectives. The general idea is to let the assertion and denial conditions governing a logical connective determine a meaning for that connective when the rules completely determine truth-conditional content. This view echoes that suggested in Peacocke (1986b), with the general requirement that:

> **General requirement**: The given rules of inference, together with an account of how the contribution to truth-conditions made by a logical constant is determined from those rules of inference, fixes the correct contribution to the truth-conditions of sentences containing the constant (Peacocke, 1993, p.172).

What follows draws heavily upon Hardegree (2005); Hjortland (2014); Humberstone (2011), and the details developed in Trafford (2014). In providing a recipe for the interaction between a sequent calculi and truth-conditions we should not talk of the truth-value of a sequent. Nonetheless, we may understand a valuation as providing a counterexample, or not, to the potential validity of a sequent in terms of whether or not truth is preserved when passing from *l.h.s* formulae to right. The idea is to let incoherent positions, and therefore, provable sequents, determine a set of valuations from the universe of possible valuations, U, over a language $\mathscr{L}$. In this way, a logic induced by a proof-system can be said to determine a valuation-space $V \subseteq U$ on $\mathscr{L}$, defined as any subset of the set of all possible valuations. First, define a logic as consisting of all provable sequent inferences.

Definition 4. For a set of formulae S in a language $\mathscr{L}$, a multiple-conclusion sequent is an ordered pair, Γ, Δ, (where $\Gamma \cup \Delta \in WFF$, and where Γ, Δ are sets of formulae of S). A multiple-conclusion logic L is an ordered pair $\langle S, L\rangle$, where L is the set of binary relations $\vdash_L$ between finite subsets of S and finite subsets of S. A rule, $R^{\#}$, defined for a logic L consists of a set of sequent premises and a set of sequent conclusions $\{SEQ_P\} \rightarrow \{SEQ_C\}$. We call the set of provable sequent in L, L-valid, such that $\Gamma \vdash \Delta =_{df} \{\langle \Gamma, \Delta\rangle \text{ is } L\text{-valid}\}$.

Definition 5. $\mathcal{V}$ is a set of truth-values, and $\mathcal{D} \subseteq \mathcal{V}$ designated values. A valuation v is a function on $\mathscr{L}$ assigning a truth-value $\in \mathcal{V}$ to a formula in S where $v : S \rightarrow \{\mathcal{V}\}$. Classically, we have $\mathcal{V} = \{1, 0\}$, and $\mathcal{D} = \{1\}$.

Before defining a relation between the two, first, note that (Definition 3) an incoherent position $[\Gamma : \Delta]$ allows that $\Gamma \vdash \Delta$, which we understood as saying that an agent is mistaken to assert all $\alpha \in \Gamma$ and reject all $\beta \in \Delta$. Equivalently, $[\Gamma : \Delta]$ is incoherent just in case some $\alpha \in \Gamma$ is denied, or some $\beta \in \Delta$ is asserted. This, and given that we read $\Gamma \vdash \Delta$ as (classically) equivalent to $\{\alpha_1 \wedge \alpha_2 \ldots . \wedge \alpha_n\} \rightarrow \{\beta_1 \vee \beta_2 \ldots . \vee \beta_n\}$, gives us that $\Gamma \vdash \Delta$ is valid just in case some $\alpha \in \Gamma$ is denied, or some $\beta \in \Delta$ is asserted.

We use this to define a relationship between sequents and valuations in terms of *satisfaction*.

Definition 6. A sequent $\Gamma \vdash \Delta$ is *satisfied* by a valuation v just in case $v(\alpha) = 0$ for some $\alpha \in \Gamma$, or $v(\beta) = 1$ for some $\alpha \in \Delta$, otherwise v *refutes* the argument.

Because we have Cut, this is easily extensible to every formula in $\mathscr{L}$.

Example 7. For a sequent $\Gamma, \alpha \vdash \Delta$ to be valid, we know that each $v \in V$ has $v(\alpha) = 0$ for some $\alpha \in \Gamma \cup \alpha$, or $v(\beta) = 1$ for some $\beta \in \Delta$. Similarly, for $\Gamma \vdash \Delta, \alpha$ to be valid, each $v \in V$ has $v(\alpha) = 0$ for some $\alpha \in \Gamma$, or $v(\beta) = 1$ for some $\beta \in \Delta \cup \alpha$.

Weakening plays the role of ensuring that each formula $\alpha \in S$ appears on either the *l.h.s* or the *r.h.s* of a sequent. Cut ensures that no formula $\alpha \in S$ appears on both. Think of this in terms of valuations. Then Identity tells us that all formulas $\alpha \in S$ takes either 1 or 0, and Cut tells us that no formula takes both. In effect, cut allows for a partition of formulas into those that take the value 1, and those that take 0.

Then, let L determine a valuation-space by determining the set of admissible valuations V that are consistent with L.

Definition 8. (L-consistency) A valuation $v \in U$ is L-consistent iff v satisfies each provable sequent in L.

Theorem 9. *A logic L determines a valuation-space V which consists of the set of valuations that are consistent with every provable sequent in L.*

Proof. A valuation space $V \subseteq U$ is determined by L when each $v \in U$ is L-consistent, so $\mathbb{V}(L) =_{df} \{v \in U : v \text{ is } L\text{-consistent}\}$. Then V consists of the set of valuations that are consistent with every provable sequent in L by definition. □

1.3 Completeness

We are not quite home and dry here, since, whilst Theorem 9 may hold for a logic, it is possible that a logic fails to determine a unique valuation space. In which case, we will not have achieved a modest inferentialist account for that logic, since the meanings of the connectives will, to some degree, be underdetermined. As is well known (Belnap and Massey, 1990; Carnap, 1943; Dunn and Hardegree, 2001; Garson, 2010; Hardegree, 2005; Hjortland, 2014; Humberstone, 2011; Shoesmith and Smiley, 1978), single-conclusion formulations of classical logic fail in this way.

We clarify this by noting that a valuation-space may also be defined semantically. On this view, classically speaking, we expect V_{CPL} to be induced by the truth-conditional interpretation of each classical connective. Then V_{CPL} contains those valuations $v : S \to \{1, 0\}$ that obey the truth-conditional clauses for the connectives of classical propositional logic. We can then use this valuation-space to determine a logic.

Definition 10. (V-validity) An sequent $\langle \Gamma, \Delta \rangle$ is V-valid iff, for all $v \in V$, v satisfies $\langle \Gamma, \Delta \rangle$.

Theorem 11. *Starting from a valuation-space, V determines a logic L w.r.t V where all the V-valid arguments are L-valid.*

Proof. Let the set of all V-valid arguments constitute the logic L, which is now determined by V: $\mathbb{L}(V) =_{df} \{\langle \Gamma, \alpha \rangle \in L : \langle \Gamma, \alpha \rangle \text{ is } V\text{-valid}\}$. All arguments valid on V are $\mathbb{L}(V)$-valid by definition. □

So, it is possible to transition between a valuation-space and a logic, and vice-versa.[1] This allows us to define an abstract completeness theorem which, following Dunn and Hardegree (2001), I call *absoluteness*.

Fact 12. *(Hardegree, 2005) For any L, V;*

- L is absolute iff $L = \mathbb{L}(\mathbb{V}(L))$
- V is absolute iff $V = \mathbb{V}(\mathbb{L}(V))$

When absoluteness holds, we have a guarantee that the determining relationship between L, V is complete. We know (Theorem 11) that a semantic structure $\langle S, V \rangle$ is determined by an inferential structure $\langle S, L \rangle$ when the set of L-valid sequents are such that V comprises the set of valuations consistent with each sequent in L. Absoluteness on V tells us that $V \subseteq U$ is the only valuation-space consistent with $\langle S, L \rangle$, and absoluteness on L tells us that L is the only set of sequents that can be associated with $\langle S, V \rangle$. So, absoluteness provides a standard by which to analyze the determining relationship between an inferential structure and a semantic structure.

Theorem 13. *(General determination theory) A semantic structure $\langle S, V \rangle$ is completely determined by an inferential structure $\langle S, L \rangle$ when L, V are absolute.*

[1]Details can be found in appendix 1.

We can utilize absoluteness to show that multiple-conclusion classical sequent logic completely determines the classical semantics.

Definition 14. (V-consistency) A valuation v is V-consistent when v satisfies every V-valid argument (Hardegree, 2005).

Lemma 15. *By the definition of absoluteness, V is absolute iff V contains every V-consistent valuation.*

Proof. See appendix 1. □

Theorem 16. *For any $V \subseteq U$, $V = \mathbb{V}(\mathbb{L}(V))$ (in the SET-SET framework).*

Proof. See appendix 1. □

An immediate corollary is that the bilateralist framework gives us a modest inferentialist account of classical logic where the valuation-space determined by the classical proof-theory uniquely determines the valuation-space V_{CPL}. Note that absoluteness does not hold for a classical proof-theory in single-conclusion format, since the valuation-space determined by the connectives $\neg, \vee$ is compatible with valuations $\notin V_{CPL}$.[2]

2 Adding Transparent Truth

One of the motivating issues in the development of paraconsistent logics is the addition of a truth predicate to classical logic. This section looks at a typical paraconsistent response that expands the set of truth-values. §2.1 shows what happens when we introduce a transparent truth-predicate into the classical system; §2.2 outlines the model theory for a multiple-conclusion paraconsistent semantics based on the logic LP^+. §2.3 goes on to show that the system fails to be absolute, which suggests that the bilateralist account can not provide an inferentialist home for paraconsistent logics.

2.1 Transparent truth

Following Beall (2009), a transparent truth predicate is a notion of truth that is "see through", such that $T(\ulcorner\alpha\urcorner)$ and α are intersubstitutable in all transparent contexts, and for all $\alpha \in S$.[3] Given Identity, this gives us familiar rules for T: $T(\ulcorner\alpha\urcorner) \vdash \alpha$; and $\alpha \vdash T(\ulcorner\alpha\urcorner)$. In terms of positions, we read this as saying for any incoherent position in which $T(\ulcorner\alpha\urcorner)$ appears, α is also incoherent. This gives us the following sequent rules:

$$\frac{\Gamma, \alpha \vdash \Delta}{\Gamma, T(\ulcorner\alpha\urcorner) \vdash \Delta}\ (T\text{-L}) \qquad\qquad \frac{\Gamma \vdash \alpha, \Delta}{\Gamma \vdash T(\ulcorner\alpha\urcorner), \Delta}\ (T\text{-R})$$

Taken alone, and given that we are working in SET-SET, both $L_\neg$, L_T are absolute. Taken together, they result in typical paradoxes. Supposing self-reference to be available in the language, we can construct a formula, Θ, of the form $\Theta : \neg T(\ulcorner\Theta\urcorner)$. In the classical framework that we are working in thus far, this quickly gets us into trouble:

[2]See appendix 1.

[3]Corner brackets indicate a name-forming device.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{T(\ulcorner\Theta\urcorner) \vdash T(\ulcorner\Theta\urcorner)}{\neg T(\ulcorner\Theta\urcorner) \vdash \neg T(\ulcorner\Theta\urcorner)}}{\Theta \vdash \Theta}}{T(\ulcorner\Theta\urcorner) \vdash \Theta}}{\vdash \neg T(\ulcorner\Theta\urcorner), \Theta}}{\vdash \Theta} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{T(\ulcorner\Theta\urcorner) \vdash T(\ulcorner\Theta\urcorner)}{\neg T(\ulcorner\Theta\urcorner) \vdash \neg T(\ulcorner\Theta\urcorner)}}{\Theta \vdash \Theta}}{\Theta \vdash T(\ulcorner\Theta\urcorner)}}{\Theta, \neg T(\ulcorner\Theta\urcorner) \vdash}}{\Theta \vdash}}{\vdash}$$

The final step uses a single application of Cut, yielding the empty sequent, which is extensible to all S. So successive application of the rules allows us to infer any conclusion from any premise. If we attempt to let $L_{\neg T}$ determine a valuation space, it will be incoherent since the only $v \in U$ that are consistent with it are $v_t =_{df} \{\alpha \in S : v(\alpha) = 1\}$, and $v_f =_{df} \{\alpha \in S : v(\alpha) = 0\}$. In other words, the addition of T to L_{CPL} entails that L_{CPL} fails to determine a coherent V, and so fails to determine the meanings of the connectives.

2.2 *LP* model theory

A typical paraconsistent response to the above employs a logic such as Graham Priest's Logic of Paradox (LP), which allows for some sentences, such as Liar sentences to be "gluts", that is, both true and false. This has been extensively argued for in the literature (Priest, 2006). This section outlines the basic model theory for LP. However, in keeping with the discussion above, we will remain within a multiple-conclusion structure, and, following Beall (2013), denote this LP^+.

We extend Definition 5 to allow three truth-values so that $\mathcal{V} = \{1, b, 0\}$, and $\mathcal{D} = \{1, b\}$. Again, we expect V to be induced by the truth-conditional interpretation of the standard connectives. Then V contains those valuations $v : S \to \{1, b, 0\}$ that obey the truth-conditional clauses for the connectives.

Definition 17. The connectives $\{\wedge, \vee, \neg\}$ satisfy the truth-conditional clauses:
(i). Conjunction: $v(\alpha \wedge \beta) = min\,\{v(\alpha), v(\beta)\}$.
(ii). Disjunction: $v(\alpha \vee \beta) = max\,\{v(\alpha), v(\beta)\}$.
(iii). Negation: $v(\neg\alpha) = 1 - v(\alpha)$

Let V_{LP} be the valuation-space comprising the set of valuations induced for each $\alpha \in S$.

Definition 18. A sequent, $\Gamma \vdash \Delta$, is refuted by a valuation v iff, when $v(\alpha) \in D$ for each $\alpha \in \Gamma$, $v(\beta) \notin D$ for all $\beta \in \Delta$); and otherwise satisfied by v.

Definition 19. A sequent is V-valid iff it is satisfied by each $v \in V$. Then;

- $\mathbb{L}(V) =_{df} \{\langle\Gamma, \alpha\rangle : \langle\Gamma, \alpha\rangle \text{ is } V\text{-valid}\}$.

Definition 20. For a set of formulae S in a language $\mathscr{L}$, an LP^+ sequent is an ordered pair, Γ, Δ, (where $\Gamma \cup \Delta \subseteq S$, and where Γ, Δ are sets of formulae of S). The logic LP^+ is an ordered pair $\langle S, LP^+\rangle$, where LP^+ is the set of binary relations $\vdash_{LP^+}$ between finite subsets of S and finite subsets of S. We call the set of provable sequent in LP^+, LP^+-valid, such that $\Gamma \vdash \Delta =_{df} \{\langle\Gamma, \Delta\rangle \text{ is } LP^+\text{-valid}\}$.

We note some features of the logic LP^+. The logical truths of LP^+ are precisely those of classical propositional logic.

Proposition 21. $\vdash_{LP^+} \alpha$ *iff* α *is a classical tautology.*

Proof. LRD. Since all classical models are also LP^+ models, this is clear. RLD. Take a valuation of LP^+, v_{LP^+} and a two-valued valuation v_{CPL} which assigns 1 to α if $v(\alpha) \in D$. We can prove by induction that, if $v_{CPL}(\alpha) = 1$ then $v_{LP^+}(\alpha) \in \{1, b\}$, and if $v_{CPL}(\alpha) = 0$ then $v_{LP^+}(\alpha) \in \{b, 0\}$. Hence, if $v_{CPL}(\alpha) = 1$, for each two-valued valuation v_{CPL} then $v_{CPL}(\alpha)$ is designated for each three-valued v_{LP^+}. □

Proposition 22. *LP^+ is paraconsistent. Both (EFQ) and disjunctive syllogism (DS) are invalid in LP^+; i.e. $\alpha \wedge \neg\alpha \nvdash_{LP^+} \beta$; $\neg\alpha, \alpha \vee \beta \nvdash_{LP^+} \beta$.*

Proof. A counterexample to both is easily given when $v(\alpha) = b$, and $v(\beta) = 0$. □

2.3 Absoluteness for LP^+

Given that we are working within a modest inferentialist framework, what we are interested in is whether or not our position structures yield a proof-theory that adequately determines LP^+ models. In other words, what we require is that, starting from position structures, we can build a logic that completely determines V_{LP^+} as we have for bilateralist classical logic. Without going into any details regarding such a proof-theory, it is simple to show that this is not possible.

Theorem 23. *For any logic L of the form SET-SET, and any $V \subseteq U$ (for which $\mathcal{V} = \{1, b, 0\}$), $V \neq \mathbb{V}(\mathbb{L}(V))$.*

Proof. See appendix 2. □

The obvious issue is that we can no longer rely upon the partitioning of formulas into those taking $v = 1$, $v = 0$. For LP^+, it is possible to construct a kind of absoluteness proof, but only by defining a partition over formulas into those that take designated values and those that do not. That is, where, $D = \{\alpha \in S : v(\alpha) \in \mathcal{D}\}$ and $D^- = \{\alpha \in S : v(\alpha) \in \mathcal{D}^-\}$, (where $\alpha \in \mathcal{D}^- =_{equiv} \alpha \notin \mathcal{D}$). But, whilst partitioning into D, D^- tells us something about consequence relations for many-valued logics (which preserve $\mathcal{D}$), it involves a loss of grasp on the semantical clauses over V_{LP^+} where we want to distinguish between designated values $\{1, b\}$.

What this means is that any logic that determines V_{LP^+} will also be consistent with the valuation-space $V_{LP^+} \cup v_b$. So that logic, however we construct it, will underdetermine the semantics of the connectives that it defines since it fails to determine a unique valuation-space. It is not too much of a stretch to say that, working from a modest inferentialist account, we will have failed to adequately determine the meanings of the connectives for any logic dealing with truth-values beyond $\{1, 0\}$. In the next section, I suggest that the problem comes down to the way in which formulas are located in the SET-SET framework. Then, by expanding bilateralist positions to trilateralist positions, we may be able to expand the locations that formulas can take in sequent arguments.

3 Expanding positions

This section provides motivation for the expansion of positions to trilateralist structures of the form $\langle \Gamma : \Theta : \Delta \rangle$, where Γ indicates the set of asserted statements, Δ the denied statements, and Θ, a set of "weakly asserted" statements. A statement α is weakly asserted when asserting $\neg\alpha$ does not rule out the content α (i.e. without also denying α). First, I clarify why LP^+ fails absoluteness by looking at the roles of Cut and Identity in determining the locations of formulas over sequent arguments. This, together with a typical paraconsistent view of assertions and denials motivates the addition of the third, intermediate, location within positions.

3.1 Partitions and cut

We clarify why absoluteness fails for many-valued semantic structures by looking at the roles played by Cut and Identity. First, recall that an ordinary SET-SET sequent $\alpha_1, \ldots, \alpha_n \vdash \beta_1, \ldots, \beta_m$ is equivalent to $\{\alpha_1 \wedge \alpha_2 \ldots \wedge \alpha_n\} \rightarrow \{\beta_1 \vee \beta_2 \ldots \vee \beta_m\}$. This, by ordinary propositional logic, is equivalent to $\neg\alpha_1 \vee \neg\alpha_2 \ldots \vee \neg\alpha_n \vee \beta_1 \vee \beta_2 \ldots \vee \beta_m$.

Definition 24. Definition 18 is equivalent to saying that a valuation v satisfies a sequent $\Gamma \vdash \Delta$ iff either one of the formulae in Γ is not designated or one of the formulas in Δ is designated. Spelling this out disjunctively over $\mathcal{V}_{LP^+}$, $\Gamma \vdash \Delta$ is V_{LP^+}-valid iff, for each $v \in V_{LP^+}$, either $v(\alpha) = 0$, or $v(\alpha) = n$ for some $\alpha \in \Gamma$, or $v(\beta) = 1$, or $v(\beta) = b$ for some $\beta \in \Delta$.

On this interpretation we can think of a sequents as having two locations, with the *l.h.s* of the sequent being the undesignated location, and the *r.h.s* the designated location. So, $\Gamma \vdash \Delta$ may be rewritten $\Gamma_{D^-} \vdash \Delta_D$.

Example 25. For a sequent $\Gamma, \alpha \vdash \Delta$ to be valid, we know that each $v \in V$ has $v(\alpha) = 0$ or $v(\alpha) = n$ for some $\alpha \in \Gamma \cup \alpha$, or $v(\beta) = 1$ or $v(\beta) = b$ for some $\beta \in \Delta$. Similarly, for $\Gamma \vdash \Delta, \alpha$ to be valid, each $v \in V$ has $v(\alpha) = 0$ or $v(\alpha) = n$ for some $\alpha \in \Gamma$, or $v(\beta) = 1$ or $v(\beta) = b$ for some $\beta \in \Delta \cup \alpha$.

As this formulation makes clear, Identity plays the role of ensuring that each formula $\alpha \in S$ appears on either the *l.h.s* or the *r.h.s* of a sequent, and Cut ensures that no formula appears on both. Think of this in terms of valuations. Then Identity tells us that all formulas take a designated value or a non-designated value, and Cut that no formula takes both. So, if, as in the case of liar-like sentences, we end up in a position where a formula Θ is forced to be located on both the *l.h.s* and the *r.h.s* of a sequent, Cut tells us that something has gone wrong. No formula can be forced to be both designated and undesignated.

Inevitably, then, over ordinary sequent structures, formulae will be partitioned into those that are designated, and those that are not, with no way of discriminating between, for example, $\{1, b\}$. Whilst the system behaves nicely for two-valued cases, it is broken by the addition of admissible values $\in \mathcal{V}$. It is for this reason that absoluteness fails for ordinary sequent inferential structures determining many-valued semantic structures. What we require then, is a way of expanding the location of formulas such that the structural rules do not ride roughshod over the finer-grained distinctions between truth-values. Then, we can expect Identity to ensure that every formula $\alpha \in S$ takes some truth-value, and Cut to ensure that no formula takes more than one.

3.2 Weak assertion

Once we have allowed that assertion is not the only game in town, the expansion of positions to accommodate a third location is fairly well motivated. Think of liar-like sentences in terms of rational commitment. The presence of liar-like sentences threatens to break down the bilateralist account since Θ is not something that we can either assert or deny. In the above presentation, Θ attempts to force an overlap between assertion and denial, which, because of the role played by Cut, is impossible. Advocates of paraconsistent approaches usually take assertion and denial to be exclusive states, so that denying α, in a sense, rules out the content of α. Nonetheless, an agent can be inferentially committed to asserting certain contradictions such that $\alpha, \neg\alpha \nvdash$. This is because asserting $\neg\alpha$ is not taken to be the the dual of denying α.[4] Asserting $\neg\Theta$ will not, then have us also deny Θ. Rather, we should assert both Θ and $\neg\Theta$. To

[4] For discussion, see Parsons (1984).

follow this line of thought, Θ would then not force an overlap between assertion and denial, but between assertion of Θ and $\neg\Theta$.

On the paraconsistent response, denying α must, therefore, be stronger than asserting $\neg\alpha$. In this respect, we might think of this in terms of an additional speech act that is weaker than assertion in that it does not cancel the content of the opposite proposition. As analogy, whilst ordinary assertion and denial behave much like "exclusion negation", weak assertion will behave much like "choice negation" in a language.[5] Think of this is in terms the grounds, or evidence, that agents have for asserting or denying a proposition α.[6] When an agent asserts α, there should be grounds in the language (given a specific context) supporting α such that α can not also be denied. But in paradoxical cases such as liar-like sentences Θ, there are grounds in the language that provide support for $\neg\Theta$, but not in such a way as to also support the denial of Θ. An agent who is rationally committed to $\neg\Theta$ is not thereby committed to the denial of Θ. In order for that to be the case, we would also require some sort of grounds for ruling out the content Θ, but this is precisely what the liar-reasoning fails to provide. So, we have reason to consider a third speech-act corresponding to cases of this kind.

Definition 26. (Weak assertion) A statement is weakly asserted when asserting $\neg\alpha$ does not rule out the content α (i.e. without also denying α).

We should, of course, be wary of introducing *ad hoc* distinctions into a theory. But, in this case, there are significant reasons for making these distinctions given that we are working in a modest inferentialist framework. Even aside from the technical problems discussed above, distinguishing between ordinary and weak assertion provides a way of understanding what an agent becomes inferentially committed to in accepting some contradictions ($\alpha, \neg\alpha \nvdash$). Some formulae, such as Θ can be weakly accepted.[7]

If this is plausible, then we have a trilateralist position structure $[\Gamma : \Theta : \Delta]$, with Γ indicating the set of asserted statements, Δ the denied statements, and Θ, the set of "weakly asserted" statements. The relation between incoherent trilateralist positions and provable sequents will be pretty much as above, except that we will have to allow for a third location in our sequent structure, and the structural rules will need to account for this distinction.

4 Sequent calculi for trilateralist positions

This section develops a sequent calculus for LP^+. We being from trilateralist positions, and draw upon n-sided sequent calculi as developed in Baaz et al. (1993a,b, 1998), Hjortland (2013), and Zach (1993) to construct a proof-theory which is absolute on V_{LP^+}. The resulting proof-theory has significant benefits in addition to the fact that it is suitable for the modest inferentialist. Primarily, it offers an account of the necessary and sufficient conditions under which certain classical valid, though paraconsistently invalid, arguments can be "recaptured".

4.1 From trilateral positions to n-sequents

We rewrite the ordinary SET-SET sequent $\Gamma_0 \vdash \Gamma_1$ as a two-sided sequent $\Gamma_0|\Gamma_1$. Whilst the indices indicate values, informally this should be read in terms of attitudes, indicating that

[5] See Tappenden (1999) for cases in which the use of "not" in a natural language context indicates the rejection of an assertion without also indicating the assertion of the negation of the relevant sentence; e. g. "Some men are not chauvinists. All of them are", "John isn't wily or crazy. He's wily and crazy".

[6] See, for example, Pagin (2012).

[7] Whether or not further distinctions amongst commitments may be warranted is left for further investigation. The obvious further distinction would be the dual of weak assertion, weak rejection, which may be thought to correspond roughly to a truth-value gap (see Remark 36).

either something in Γ_0 is rejected, or something in Γ_1, accepted. Then, we simply carry over from Definition 26:

Definition 27. $\Gamma_0|\Gamma_1$ is V-valid iff either $v(\alpha) = 0$ for some $\alpha \in \Gamma_0$, or $v(\beta) = 1$ for some $\beta \in \Gamma_1$.

The idea is to expand this strategy for three formula locations using n-sided sequents.

Definition 28. An n-sided sequent Γ is an ordered n-tuple of finite sequences $\Gamma_1|\ldots|\Gamma_n$where Γ_n is the n-th component of Γ.

Where Γ is a sequent, Γ_i denotes the i-th component of the sequent, with the sequent interpreted as a disjunction of statements saying that a particular formula takes a particular location in the structure of the sequent. Then, it is straightforward to define a three-sided sequent corresponding to the logic LP^+.

Example 29. A three-sided sequent for L_{LP^+} is written as:

$$\Gamma_1|\Gamma_b|\Gamma_0$$

again, read disjunctively as saying that either $v(\alpha) = 1$ for some $\alpha \in \Gamma_1$, or $v(\theta) = b$ for some $\theta \in \Gamma_b$, or $v(\beta) = 0$ for some $\beta \in \Gamma_0$.

The relation between incoherent trilateral positions and sequent provability can now be characterized.

Definition 30. (Three-sided sequent provability) If $[\Gamma : \Theta : \Delta]$ is incoherent, then $\Gamma_1|\Gamma_b|\Gamma_0$ is valid.

We need to spell out incoherence disjunctively. A position $[\Gamma : \Theta : \Delta]$ is incoherent if an agent either denies or weakly asserts some $\alpha \in \Gamma$, or asserts or denies some $\theta \in \Theta$, or asserts or weakly asserts some $\beta \in \Delta$. We use this to define the logic LP^+.

Definition 31. A three-sided logic is an ordered pair $\langle S, L\rangle$, where L is a set of relations between finite sets of three-sided sequents of S, and each sequent argument in L is called L-valid. For the set of truth-values $\mathcal{V} = \{1, b, 0\}$, and for each location Γ_i (which is a possibly empty set of formulae $\in S$), a valuation v satisfies a three-sided sequent iff for some Γ_i, when $i \in \{1, b, 0\}$, and some formula $\alpha \in \Gamma_i, v(\alpha) = i$.

Definition 32. An three-sided sequent is V-valid iff it is satisfied by each $v \in V$. A valuation $v \in U$ is L-consistent iff v satisfies each provable sequent in L.

- $\mathbb{L}(V) =_{df} \{\langle\Gamma_1|\Gamma_b|\Gamma_0\rangle \in L : \langle\Gamma_1|\Gamma_b|\Gamma_0\rangle$ is V-valid$\}$
- $\mathbb{V}(L) =_{df} \{v \in U : v$ is L-consistent$\}$

By considering locations in trilateral positions pair-wise, we are able to ensure that the structural rules play the required roles, now formulated as follows (Baaz et al., 1993a).

$$\alpha|\alpha|\alpha(\text{Identity})$$

For each sequent location i:

$$\frac{\Gamma}{\Gamma, [i : \alpha]}\ (\text{Weakening})$$

For each couple of truth-values where $v_i \neq v_j$:

$$\frac{\Gamma, [i : \alpha] \qquad \Delta, [j : \alpha]}{\Gamma, \Delta} \ (\mathrm{Cut}(i, j))$$

Since the structure has more than two locations for formulae, Identity now makes sure that each formula takes a valuation, and Cut operates on pairs of truth-values. So, Cut still partitions formulae by ensuring that each formula takes a single truth-value only. For any $v_i \neq v_j$ for a formula A such that $v(\alpha) = v_i = v_j$, α is removed:

$$\frac{\Gamma_1|\ldots|\Gamma_i, \alpha|\ldots|\Gamma_n \qquad \Delta_1|\ldots|\Delta_j, \alpha|\ldots|\Delta_n}{\Gamma_1, \Delta_1|\ldots|\Gamma_n\Delta_n}$$

The difference is that for ordinary sequents, we had only two values to worry about, $\{1, 0\}$ (or two locations).

4.2 Proof-theory for LP^+

We construct an 3-sided proof-system in a uniform way (Baaz et al., 1993a). The structural rules remain as above and operational rules for each connective are given for each location in the three-sided sequent as follows:

$$\frac{\Gamma_0|\Gamma_b|\Gamma_1, \alpha \qquad \Gamma_0|\Gamma_b|\Gamma_1, \beta}{\Gamma_0, |\Gamma_b|\Gamma_1, \alpha \wedge \beta} \ (\wedge 1) \qquad\qquad \frac{\Gamma_0, \alpha, \beta|\Gamma_b|\Gamma_1}{\Gamma_0, \alpha \wedge \beta|\Gamma_b|\Gamma_1} \ (\wedge 0)$$

$$\frac{\Gamma_0, \alpha|\Gamma_b, \alpha|\Gamma_1 \qquad \Gamma_0, \beta|\Gamma_b, \beta|\Gamma_1 \qquad \Gamma_0|\Gamma_b, \alpha, \beta|\Gamma_1}{\Gamma_0, |\Gamma_b, \alpha \wedge \beta|\Gamma_1} \ (\wedge b)$$

$$\frac{\Gamma_0, \alpha|\Gamma_b|\Gamma_1 \qquad \Gamma_0, \beta|\Gamma_b|\Gamma_1}{\Gamma_0, \alpha \vee \beta|\Gamma_b|\Gamma_1} \ (\vee 0) \qquad\qquad \frac{\Gamma_0|\Gamma_b|\Gamma_1 \alpha, \beta}{\Gamma_0|\Gamma_b|\Gamma_1, \alpha \vee \beta} \ (\vee 1)$$

$$\frac{\Gamma_0, \alpha|\Gamma_b, \alpha|\Gamma_1 \qquad \Gamma_0, \beta|\Gamma_b, \beta|\Gamma_1 \qquad \Gamma_0|\Gamma_b, \alpha, \beta|\Gamma_1}{\Gamma_0|\Gamma_b, \alpha \vee \beta|\Gamma_1} \ (\vee b)$$

$$\frac{\Gamma_0|\Gamma_b|\Gamma_1, \alpha}{\Gamma_0, \neg\alpha|\Gamma_b|\Gamma_1} \ (\neg 0) \qquad\qquad \frac{\Gamma_0, \alpha|\Gamma_b|\Gamma_1}{\Gamma_0|\Gamma_b|\Gamma_1, \neg\alpha} \ (\neg 1)$$

$$\frac{\Gamma_0|\Gamma_b, \alpha|\Gamma_1}{\Gamma_0|\Gamma_b, \neg\alpha|\Gamma_1} \ (\neg b)$$

The logic L_{LP^+} comprises the set of valid sequent arguments determined by the proof-system. For some $V \subseteq U$ (where U has $\mathcal{V} = \{1, b, 0\}$), $V_{L_{LP^+}} = \mathbb{V}(L_{LP^+}) = \{v : v \text{ is } L_{LP^+}\text{-consistent}\}$.

Soundness and completeness for $\mathbb{V}(L_{LP^+})$ are relatively simple. First, soundness can be proved, in the usual way, by induction over proofs, since the operational rules preserve validity by definition, and the structural rules are valid. For completeness, if a sequent is V_{LP^+}-valid, then it is provable in the three-sided construction without cuts.[8] The proof can be carried over from Baaz et al. (1993b), which uses the method of reduction trees.[9]

Most importantly for our purposes, L_{LP^+} is absolute *w.r.t* V_{LP^+} as outlined semantically in §2.2.

Theorem 33. *In general, for a valuation space $V \subseteq U$, $V = \mathbb{V}(\mathbb{L}(V))$ (Hjortland, 2014).*

Proof. See appendix 3. □

[8] I note a complication regarding cut-elimination below.

[9] Priest (2008) uses the tableau system to give a completeness proof for FDE, which, though not constructed using n-sequents, can be carried over with a little tweaking.

As an immediate corollary, the 3-sided proof-theory for LP^+ yields an absoluteness result. Then, we have a modest inferentialist account that, beginning with trilateralist positions, provides a proof-theory that uniquely determines the valuation-space V_{LP^+}.

4.3 Notable features of the proof-theory

4.3.1 Finer-grained distinctions

An immediate advantage of the 3-sided proof-theory (in addition to its inferentialist scruples), is that it offers a way of maintaining fine-grained distinctions across sequent arguments. By way of illustration, we can see that there is, despite absoluteness, no way of retaining these distinctions in the ordinary sequent set-up.

Example 34. For $\mathbb{L}_{LP^+}(V_{LP^+})$, typically, we will say that $\Gamma \vdash \Delta$ is valid iff when $v(\alpha) \in D$ for each $\alpha \in \Gamma$, $v(\beta) \in D$ for some $\beta \in \Delta$, or equivalently, either $v(\alpha) \notin D$ for some $\alpha \in \Gamma$ or $v(\beta) \in D$ for some $\beta \in \Delta$. We can spell out the latter by saying that either $v(\alpha) \neq 1$ and $v(\alpha) \neq b$ for some $\alpha \in \Gamma$ or $v(\beta) = 1$ or $v(\beta) = b$ for some $\beta \in \Delta$. But, as we saw in §3.2, as long as $\vdash_{LP^+}$ obeys Cut (ensuring transitivity for the two-sided system), the differentiation amongst D is lost. This is clearer when we consider that by the definitions of V_{LP^+} and V_{LP^+}-validity, $\Gamma \vdash_{LP^+} \Delta$ when $\Gamma|\Gamma|\Delta$ is V_{LP^+}-valid. In other words, we can switch back from $\Gamma_0|\Gamma_b|\Gamma_0$ to the two-sided $\Gamma_0 \rightarrow \Gamma_b \cup \Gamma_1$ where the latter is V_{LP^+}-valid when $v(\alpha) \notin D$ for some $\alpha \in \Gamma_0$ or $v(\beta) \in D$ for some $\beta \in \Gamma_b \cup \Gamma_1$. But, there is no route back from a V_{LP^+}-valid sequent in this two-sided incarnation to a specific V-valid 3-sided sequent. In essence, this is due to the fact that, whilst Cut holds for each of the three-locations in the three-sided structure, the three-sided version of Cut does not guarantee transitivity in two-sided systems because of the way in which the three-sided locations overlap in the two-sided structure.

This provides additional reason to think that pursuing issues related to paraconsistent logics in three-sided constructions may be preferable to standard proof-theories (including those in SET-SET) because of the fine-grained nature preserved by the logic.

4.3.2 Transparent truth

Take L_{LP}, and additionally define operational rules for T:

$$\frac{\Gamma_0, \alpha|\Gamma_b|\Gamma_1}{\Gamma_0, T(\ulcorner\alpha\urcorner)|\Gamma_b|\Gamma_1}\ (T0)$$

$$\frac{\Gamma_0|\Gamma_b, \alpha|\Gamma_1}{\Gamma_0|\Gamma_b, T(\ulcorner\alpha\urcorner)|\Gamma_1}\ (Tb) \qquad \frac{\Gamma_0|\Gamma_b|\Gamma_1, \alpha}{\Gamma_0|\Gamma_b|\Gamma_1, T(\ulcorner\alpha\urcorner)}\ (T1)$$

Now, cut is no longer eliminable and the subformula property fails since T applies to formulae of any complexity, including the liar sentence. It is not clear whether or not this is problematic. For example, above, it was suggested both that $\mathcal{T}$ may be considered part of the basic set of constraints on rational commitment, and it is required for absoluteness proofs. As such, eliminating cut, whilst of technical interest, loses some philosophical motivation, particularly given that any calculus involving T suffers from a loss of the subformula property in any case.[10]

[10]It may be possible to rectify this by considering sequents in terms of multisets rather than sets, and develop a construction without the structural rule of contraction.

4.3.3 Classical recapture

As resultant from this finer-grained distinctions between formulas over sequent arguments, the three-sided proof-theory also provides a nice way of understanding "classical recapture". LP^+ is notoriously weak in comparison with classical logic. As above, both EFQ and DS are invalid in LP^+. The failure of such inferences is counter-intuitive, and it threatens to undermine significant features of ordinary reasoning. Resultantly, paraconsistent logics have come under significant criticism (e. g. Parsons 1984; discussed in Priest 2006). By way of response, various forms by which classical reasoning can be "recaptured" have been suggested (Priest, 2006, §8), where:

> **Methodological maxim:** Unless we have specific grounds for believing that the crucial conditions in a piece of quasi-valid reasoning are gluts, we may accept that reasoning. (116)

Whilst the maxim is certainly plausible, working out the logical details underlying it has proven difficult. For example, we can not force consistency by somehow adding consistency to the premise set. A natural thought would be to employ the truth-predicate, with a formulation of the law of non-contradiction, such as $\neg T(\alpha \wedge \neg\alpha)$ to "force" the consistency of truth. But, $\neg T(\alpha \wedge \neg\alpha)$ does not logically rule out the possibility that $T(\alpha \wedge \neg\alpha)$. A prominent suggestion (Priest, 2006) has been to add a stronger than material conditional to the stock of logical connectives, where, for all α, $(\alpha \wedge \neg\alpha) \rightarrow\perp$. But, arguably, this is too strong as it requires a logical connection between antecedent and consequent derived from relevant logic (Beall, 2012).

In any case, the three-sided proof-theory above has a significant advantage over these suggestions in that it provides us with necessary and sufficient conditions for when certain classically valid inferences are provable, and when they are not. For example, take the negation rules. $\neg 0$ and $\neg 1$ give the conditions for negation in L_{LP^+} corresponding to assertion and denial. A formula may be denied if its negation is asserted, and asserted if its negation is denied. This corresponds to the classical rules $\neg$-L or $\neg$-R. The difference (and the reason why $\neg$-L and $\neg$-R no longer force the equivalence of asserting $\neg\alpha$ with denying α) is that we also have the rule $\neg$b. This gives the conditions for negation corresponding to weak assertion. The negation of a formula may be weakly asserted when the formula is weakly asserted. Taken together, the rules impose constraints on $\mathbb{V}(L_{LP^+})$ corresponding to the semantic definition of negation (Definition 17).

Example 35. Take disjunctive syllogism (DS) as example. As we saw in Proposition 22, (DS) is invalid in LP^+. Let us analyze this in detail in the three-sided logic. First, consider the disjunction rule $\vee b$ (since $\vee 0$ and $\vee 1$ are familiar). The rule provides three (necessary and sufficient) conditions under which a disjunction $\alpha \vee \beta$ may be weakly asserted: α or β is weakly asserted; and α is denied or weakly asserted; and β is denied or weakly asserted. As before, we have a counterexample to (DS) where α is weakly asserted, and β denied. Since α is weakly asserted, we know that $\neg\alpha$ is weakly asserted. Then, we can derive the provable sequent argument $\Gamma, \neg\alpha, \alpha \vee \beta \vdash \alpha, \neg\alpha, \Delta$:

$$\dfrac{\dfrac{\Gamma_0|\Gamma_b, \alpha \vee \beta|\Gamma_1}{\Gamma_0, \beta|\Gamma_b, \alpha|\Gamma_1}(\vee b) \qquad \Gamma_0|\Gamma_b, \neg\alpha|\Gamma_1}{\Gamma_0|\Gamma_b, \alpha, \neg\alpha|\Gamma_1}(\neg b)$$

But, notice that we also have the resources in the proof-system providing the necessary and sufficient conditions for when it is permissible to derive (DS). When, for example, α and β are asserted, we know (by the rule $\vee 1$) that $\alpha \vee \beta$ may be asserted. We also know (by $\neg 1$) that whenever α is asserted, $\neg\alpha$ may be denied. This allows us to derive (DS) $(\Gamma, \neg\alpha, \alpha \vee \beta \vdash \beta, \Delta)$:

$$\dfrac{\dfrac{\Gamma_0|\Gamma_b|\Gamma_1, \alpha \vee \beta}{\Gamma_0|\Gamma_b|\Gamma_1, \alpha, \beta}\,(\vee 1) \qquad \dfrac{\Gamma_0|\Gamma_b|\Gamma_1, \neg\alpha}{\Gamma_0, \alpha|\Gamma_b|\Gamma_1}\,(\neg 1)}{\Gamma_0|\Gamma_b|\Gamma_1, \beta}\,(\text{Cut})$$

Remark 36. It is not difficult to see that the 3-sided construction for LP is analogous to a construction for the Kleene 3-valued logic, K_3. K_3 similarly has three algebraic truth-values, with the middle value typically denoted i for indeterminate, so $\mathcal{V} = \{1, i, 0\}$. In distinction with LP, K_3 has $\mathcal{D} = \{1\}$. It is well known that K_3 is paracomplete (law of excluded middle may not hold), whereas LP is paraconsistent. So, the two logics have distinct consequence relations, since, whilst they share the same interpretation of standard connectives, they differ with respect to the interpretation of the truth-values. This fact is typically reflected in standard proof-theoretic constructions of the two logics.[11] However, in an n-sequent construction, the two coincide apart from the decoration of the middle sequent. Nonetheless, whilst the decoration is arbitrary, it reflects a distinction between the two structures at the level of provability.[12] By the translation in Example 34, we say that $\Gamma \vdash_{LP} \Delta$, whenever $\Gamma|\Delta|\Delta$ is derivable in L_{LP}; in distinction, $\Gamma \vdash_{K_3} \Delta$, whenever $\Gamma|\Gamma|\Delta$ is derivable in L_{K_3}, where L_{K_3} is equivalent to L_{LP} (just decorate the middle sequent with i in place of b). This difference allows us to distinguish between the two structures, so that, for example, law of excluded middle is derivable in L_{LP}, but not in L_{K_3}. Additionally, we know, by Theorem 33, that the semantics for L_{K_3}, $\mathbb{V}(L_{K_3})$, will be absolute, and, since the designated values differ from that of $\mathbb{V}(L_{LP})$, the consequence relation differs accordingly.

We can make sense of this approach to L_{K_3} by considering the dual to weak assertion, "weak denial", where a statement α is weakly denied when denying $\neg\alpha$ does not rule in the content α (i.e. without also asserting α). This is typical in discussions of paracomplete logics. Of course, if we allow weak denial alongside weak assertion, then it is possible to construct a four-sided sequent structure by simply expanding sequents to: $\Gamma_1|\Gamma_b|\Gamma_i|\Gamma_0$. Unsurprisingly, the semantic structure determined by the full (four-sided) sequent structure is precisely that of first degree entailment (FDE), and, again, by Theorem 34, their relationship is absolute.

Conclusion

This paper has developed a modest inferentialist approach to paradox. Starting with a bilateralist framework, an account of absoluteness for multiple-conclusion classical logic was provided. This, however, does not carry over to many-valued logics such as LP^+, primarily due to structural deficiencies in both bilateral positions and ordinary sequent formulas. By way of response, I suggested expanding bilateralism to trilateralism, which incorporates a third, intermediate, speech-act, "weak assertion". In doing so, we can construct, out of trilateral positions, a proof-theory using three-sided sequents. This proof-theory for LP^+ is absolute, and also provides a simple way of understanding classical recapture.

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[11] See, for example, Priest (2008).

[12] See, for example, Hjortland (2013), where the relation between the two structures is discussed in the context of logical pluralism.

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Appendices

Appendix 1

The below follows the account given in Trafford (2014).

Thinking of L and $\mathbb{V}(L)$ in the abstract (as not yet constrained by any proof-system), we have, in effect, two partially ordered sets defined over $\mathscr{L}$ (Hardegree, 2005):

(P1) The set of all valuation-spaces $V \subseteq U$ on $\mathscr{L}$, ordered by set-inclusion;

(P2) The set of all logics $L \subseteq L'$ on $\mathscr{L}$, ordered by set-inclusion.

The relation between the two induces an antitone Galois connection between valuation spaces and logics (Dunn and Hardegree, 2001; Hardegree, 2005; Hjortland, 2014; Humberstone, 2011), where a generalized Galois connection is an adjunction of maps between partially ordered sets in terms of order preservation functions.

Definition. A Galois connection between posets P, Q is a map: $f_1 : P \to Q$ and $f_2 : Q \to P$ where the following conditions hold for all subsets P_n, Q_n of P, Q:

$$P_0 \subseteq f_2(f_1(P_0)) \tag{4.1}$$

$$Q_0 \subseteq f_1(f_2(Q_0)) \tag{4.2}$$

$$P_0 \subseteq P_1 \Rightarrow f_1(P_1) \subseteq f_1(P_0) \tag{4.3}$$

$$T_0 \subseteq T_1 \Rightarrow f_2(T_1) \subseteq f_2(T_0) \tag{4.4}$$

It follows that $f_1 \subseteq f_1 f_2 f_1 \subseteq f_1$, so $f_1 = f_1 f_2 f_1$ and also $f_2 = f_2 f_1 f_2$.

For our purposes, here P is the set of all valuations over $\mathscr{L}$, and Q the set of all sequents in L, with satisfaction being the relation defining the functions between them. For any valuation space V, $f_1(V)$ consists of the set of sequents satisfied by each $v \in V$, i.e. $f_1(V) = \mathbb{L}(V)$. For any logic L, $f_2(L)$ will consist of the set of valuations that satisfy every sequent in L, i.e. $f_2(L) = \mathbb{V}(L)$.

With this, we can define a closure operator cl as a function on the posets $\langle V, L\rangle$, given that cl obeys the following clauses for all x, y on $\langle V, L\rangle$:

(c1) $x \leq cl(x)$

(c2) $cl(cl(x)) \leq cl(x)$

(c3) $x \leq y \to cl(x) \leq cl(y)$

This ensures that, where cl is a closure operator on a poset $\langle P, \leq\rangle$, and x is an element of P, then x is closed iff $cl(x) = x$. In our context, this gives us an abstract completeness theorem over $\langle \mathbb{V}, \mathbb{L}\rangle$.

Fact. *For each $V \subseteq U$ and $L \subseteq L'$ (for some S):*

$$L \subseteq \mathbb{L}(\mathbb{V}(L)) \tag{4.5}$$

$$V \subseteq \mathbb{V}(\mathbb{L}(V)) \tag{4.6}$$

$$L \subseteq L' \Rightarrow \mathbb{V}(L') \subseteq \mathbb{V}(L) \tag{4.7}$$

$$V \subseteq U \Rightarrow \mathbb{L}(U) \subseteq \mathbb{L}(V) \tag{4.8}$$

Proof. Given at length in Hardegree (2005). □

With this, we define *absoluteness*.

Fact. *(Hardegree, 2005) For any L, V;*

- L is absolute iff $L = \mathbb{L}(\mathbb{V}(L))$
- V is absolute iff $V = \mathbb{V}(\mathbb{L}(V))$.

Proof. By the fact that L, V form a Galois map, and the definition of Galois closure (c1–3). □

Resultantly, we can give general soundness and completeness theorems for the construction of any normal, finite logic.

Fact. *Let Γ be any set of formulas in S. Define v_Γ, as: $v_\Gamma(\alpha) = 1$ if $\Gamma \vdash \alpha$, and $v_\Gamma(\alpha) = 0$ otherwise. Then v_Γ is L-consistent and $v_\Gamma \in \mathbb{V}(L)$.*

Proof. (Hardegree, 2005) If not, there must be a sequent, $\Delta \vdash \beta$ in L that is refuted by v_Γ, so that $v_\Gamma(\Delta) = 1$ and $v_\Gamma(\beta) = 0$. Given the way in which v_Γ is defined, this means that $\Gamma \vdash \Delta$. But, $\Delta \vdash \beta$ is L-valid, and given that the $\vdash$ associated with L is closed under transitivity, it follows that $\Gamma \vdash \beta$, so by the definition of v_Γ, $v_\Gamma(\beta) = 1$, so v_Γ does not refute $\Gamma \vdash \beta$. □

Fact. *In general, for any finite normal logic L, $L = \mathbb{L}(\mathbb{V}(L))$.*

Proof. (Hardegree, 2005) Suppose that some $\langle \Gamma \vdash \beta \rangle \notin L$, to show that $\langle \Gamma \vdash \beta \rangle \notin \mathbb{L}(\mathbb{V}(L))$ (in other words, it is refuted by $\mathbb{V}(L)$. Take the valuation v_Γ, which by Lemma 17 is in $\mathbb{V}(L)$. By definition, v_Γ satisfies all derivable sequents of L. Since L is reflexive, each element of Γ is derivable in L, so v_Γ satisfies Γ. But, since $\langle \Gamma \vdash \beta \rangle$ is not L-valid, $\beta \notin \Gamma$, so v_Γ refutes β. Then v_Γ refutes $\langle \Gamma \vdash \beta \rangle$, and so too does $\mathbb{V}(L)$, thus $\langle \Gamma \vdash \beta \rangle \notin \mathbb{L}(\mathbb{V}(L))$. □

Theorem. *For any $V \subseteq U$, $V = \mathbb{V}(\mathbb{L}(V))$.*

Proof. (Dunn and Hardegree, 2001, p. 200) We prove contra-positively by defining a valuation $v_0 \notin V$ (in order to show that $v_0 \notin \mathbb{V}(\mathbb{L}(V))$). Then define $T = \{\alpha \in S : v_0(\alpha) = 1\}$ and $F = \{\alpha \in S : v_0(\alpha) = 0\}$. For any $v \in V$, $v \neq v_0$, so either $v(\alpha) = 0$ for some $\alpha \in T$ or $v(\alpha) = 1$ for some $\alpha \in F$. Then v satisfies $T \vdash F$, and it follows that $T \vdash F$ is valid on V. But, by definition, v_0 refutes $T \vdash F$, so $v_0 \notin \mathbb{V}(\mathbb{L}(V))$. □

Example. Absoluteness does not hold for V_{CPL} because the classical proof-system defining the connectives $\neg, \vee$ is compatible with valuations $\notin V_{CPL}$. For example, say we define negation in this framework as:

$$\frac{\Gamma, A \vdash B \wedge \neg B}{\Gamma \vdash \neg A}\,(Reductio) \qquad\qquad \frac{\Gamma \vdash A \qquad \Gamma \vdash \neg A}{\Gamma \vdash B}\,(EFQ)$$

>From this, induce $L_\neg$, and determine a corresponding valuation space $V_\neg$. Then $V_\neg \neq V_{CPL}$ if the latter is supposed accord with the truth-functional definition $f^\neg$:

$$f^{\neg}(x) = \begin{cases} 1 & if\ x = 0 \\ 0 & if\ x = 1 \end{cases}$$

More precisely, $\mathbb{V} \neq \mathbb{V}(\mathbb{L}(V_{CPL\neg}))$ because, whilst $L_\neg$ is sound and complete *w.r.t* V_{CPL}, it is also sound and complete *w.r.t* alternative semantic structures. For example, Hardegree (2005) defines a super-valuation associated with a valuation-space V to be the valuation v_V where, for every WFF, α, $v_V(\alpha) = 1$ if $v(\alpha) = 1$ for every $v \in V$ and $v_V(\alpha) = 0$ otherwise. Clearly, $L_\neg$ is sound and complete *w.r.t* both V_{CPL}, and $V_{CPL} \cup v_V$.

Appendix 2

Theorem. *For any logic L of the form SET-SET, and any* $V \subseteq U$ *(for which* $\mathcal{V} = \{1, b, 0\}$*),* $V \neq \mathbb{V}(\mathbb{L}(V))$.

Proof. We show that this is the case by showing that the absoluteness proof when $\mathcal{V} = \{1, 0\}$ (Theorem 16) is inadequate when we add intermediate truth-values. First, we define a valuation $v_b \notin V_{LP^+}$ (in order to show that $v_b \in \mathbb{V}(\mathbb{L}(V_{LP^+}))$). Let v_b be defined such that, for every $\alpha \in S$, $v_b(\alpha) = b$ (clearly, $v_b \notin V_{LP^+}$ since v_b makes everything a glut at once). Since $v_b \notin V_{LP^+}$, for each $v \in V_{LP^+}$, there is some formula α for which $v(\alpha) \neq v_b(\alpha)$. For such a formula, let $\Gamma = \{\alpha \in S : v(\alpha) \notin \mathcal{D}\}$ and $\Delta = \{\alpha \in S : v(\alpha) \in \mathcal{D}\}$. Keep in mind that $v_b(\Gamma) = b$, and $v_b(\Delta) = b$. Where $L = \mathbb{L}(V_{LP^+})$, $\Gamma \vdash \Delta$, since either $\alpha \in \Gamma$, and so $v(\alpha) \notin \mathcal{D}$, or $\alpha \in \Delta$, and so $v(\alpha) \in \mathcal{D}$. However, unlike the case where $\mathcal{V} = \{1, 0\}$, v_b also satisfies $\Gamma \vdash \Delta$, so we can not use that partition to rid ourselves of inadmissible valuations. □

Appendix 3

Theorem. *In general, for a valuation space* $V \subseteq U$, $V = \mathbb{V}(\mathbb{L}(V))$ *(Hjortland, 2014).*

Proof. We proceed by supposing that $v_0 \notin V$, with the intent to show that $v_0 \notin \mathbb{V}(\mathbb{L}(V))$. First, define $\Gamma_1 = \{A \in WFF : v_0(A) \neq v_1\}$; $\Gamma_2 = \{A \in WFF : v_0(A) \neq v_2\}$; ... ; $\Gamma_n = \{A \in WFF : v_0(A) \neq v_n\}$. As is clear, for each $v \neq v_0$, v satisfies the sequent $\Gamma_1|\Gamma_2|\ldots|\Gamma_n$, since $v \neq v_0$, there is a formula A where $v(A) \neq v_0(A)$. If we assume that $v_0(A) = v_i$, then, for some $j \neq i$, $v(A) = v_j$. By definition, $A \in \Gamma_k$ for each Γ_k where $k \neq i$, so $A \in \Gamma_j$. Hence v satisfies $\Gamma_1|\Gamma_2|\ldots|\Gamma_n$. However, v_0 fails to satisfy $\Gamma_1|\Gamma_2|\ldots|\Gamma_n$, since if it did, then for some i there is a formula $A \in \Gamma_i$ such that $v_0(A) = v_i$. But, by definition, if $A \in \Gamma_i$, then $v_0(A) = v_i$. Hence, $\Gamma_1|\Gamma_2|\ldots|\Gamma_n$ is V-valid since it is satisfied by each $v \neq v_0$, and, because it fails to be satisfied by v_0, $v_0 \notin \mathbb{V}(\mathbb{L}(V))$. □